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JHEP07(2009)035

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Non uniform black strings and critical dimensions in AdS_d

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ABSTRACT: We study the equations of black strings in spacetimes of arbitrary dimensions with a negative cosmological constant and construct numerically non uniform black string solutions. Our results suggest the existence of a localised black hole in asymptotically locally AdS spacetime. We also present evidences for a dependence of the critical dimension on the horizon radius. The critical dimension represents the dimension where the order of the phase transition between uniform and non uniform black string changes. Finally, we argue that both, the regular asymptotically locally AdS solution and AdS black string solutions with a very small horizon radius, present a negative tension. This turns out to be an unexpected feature of the solutions.

KEYWORDS: Classical Theories of Gravity, Field Theories in Higher Dimensions, Black Holes

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1 Introduction

The last decade has witnessed a growing interest for solutions in AdS spacetime. Part of the reason is the *AdS/CFT* [1, 2] duality in string theory. This duality relates classical AdS gravity solutions to operators in a conformal field theory defined in the background of the boundary AdS spacetime. Among others, an application of *AdS/CFT* duality allowed to interpret the transition between the thermal AdS spacetime and the AdS black hole as a transition from a confining to a deconfining phase in the dual CFT [3]. At some regime in the dual *CFT*, the theory is approximated by a relativistic fluid dynamic theory. Many works have been recently carried out in order to investigate different fluids configurations. This hydrodynamical approach provides information on the expected properties of the solutions on the gravity side (see for example [4, 5]). It is clear that AdS spacetimes are interesting from the *AdS/CFT* point of view, but they are also interesting by themselves, since unexpected properties can arise, due to the AdS curvature effect [6]. The duality is better understood in five dimensions, where the AdS_5 space is dual to a $\mathcal{N} = 4$ Super Yang-Mills theory. Clearly, *AdS/CFT* and even more, string theory involves higher dimensional models for gravity. But it turned out that in more than four spacetime dimensions and without a cosmological constant, various theorems known to hold in four

dimensions are not true anymore. For instance, the uniqueness theorem doesn't hold anymore in higher dimensional gravity, where different black hole solutions for a given set of physical parameters can exist as it is the case for the Myers-Perry black hole and the black rings [7]. In four dimension, the uniqueness theorem imposes S_2 horizon topology, while in higher dimensions, apart from the black hole solutions with horizon topology S_{d-2} [8], one can construct black strings solutions with horizon topology $S_{d-3} \times S_1$ [9] but also solutions with toroidal horizon topology [7]. All these can be generalised to charged, rotating and/or (A)dS spacetimes [10–21] (see also [22] for a review).

However, the black string solutions are known to suffer from a long wavelength instability, namely the Gregory-Laflamme instability [23]. This instability is somehow the counterpart of the Plateau-Rayleigh instability in fluid dynamics [24]. After that discovery in '93, it was widely believed that the black string would decay to a localised black hole, but it has been argued that such a phase transition would take an infinite proper time at the horizon [25]. This lead to postulate the existence of an intermediate solution, the non uniform black string, which has been constructed first in a perturbation theory [26] then in the full non linear regime [27].

More recently, it has been shown that the Gregory-Laflamme instability persists in asymptotically locally AdS spacetimes [28], at least for small AdS black strings where the horizon radius is small compared to the AdS curvature, and AdS non uniform black strings have been constructed perturbatively in various dimensions¹ [6, 29]. In [6], partial results were presented for the non perturbative non uniform black string solution in AdS_6 .

The aim of this paper is to extend the preliminary results obtained in [6] to an arbitrary number of dimension and to provide a more detailed study of non uniform black string solutions in asymptotically locally AdS spacetimes. The paper is organised as follows: first we present the model, the equations and boundary conditions for the non uniform black string problem in section 2. In section 3, we give the asymptotic and near horizon solution for the non uniform black strings, as well as the thermodynamical properties of the solutions. Section 3 also includes a discussion on the sign of the tension for small AdS black strings, as well as a discussion on the entropy comparison between uniform solutions and non uniform solutions with the same mass. We describe the numerical method and results in section 4. It is well known that in asymptotically locally flat spacetime the order of the phase transition between the uniform and non uniform phase changes from one to higher above a critical dimension [30], but we will argue that this property changes drastically in asymptotically locally AdS spacetime. All our results are summarised in the concluding section 5.

¹It should be reminded that black strings in AdS admit two phases: small AdS black strings, which are thermodynamically and dynamically instable and big AdS black strings which are thermodynamically and dynamically stable [20, 28].

2 The model

We consider the d -dimensional Einstein-Hilbert action with the presence of a negative cosmological constant Λ , supplemented by the Hawking Gibbons boundary term and with:

$$S = \frac{1}{16\pi} \int_{\mathcal{M}} \sqrt{-g} \left(R + \frac{(d-1)(d-2)}{\ell^2} \right) d^d x - \frac{1}{8\pi} \int_{\partial\mathcal{M}} \sqrt{-h} K d^{d-1} x \quad (2.1)$$

in units where the Newton constant equals one, $G_d = 1$ and where \mathcal{M} is the manifold under consideration, $\partial\mathcal{M}$ is the boundary of the manifold, g is the determinant of the metric, R is the scalar curvature, ℓ is the AdS radius related to the cosmological constant according to $\Lambda = -(d-1)(d-2)/(2\ell^2)$, h is the determinant of the induced metric on the boundary and K is the trace of the extrinsic curvature of the boundary manifold.

In order to construct the non uniform strings, we supplement the action (2.1) by the following ansatz

$$ds^2 = -b(r)e^{2A(r,z)} dt^2 + e^{2B(r,z)} \left(\frac{dr^2}{f(r)} + a(r) dz^2 \right) + r^2 e^{2C(r,z)} d\Omega_{d-3}^2 \quad (2.2)$$

where a, b, f refer to the background functions, solution to the uniform black string problem (where $A = B = C = 0$) [20] and A, B, C are the non uniform corrections to the background. The functions a, b, f behave like r^2/ℓ^2 for large r :

$$a(r) \approx b(r) \approx f(r) \approx \frac{r^2}{\ell^2} + \mathcal{O}(0), \quad (2.3)$$

describing an asymptotically locally AdS spacetime.

However, the solutions are a priori independent of the a, b functions normalisation; this means that the functions a, b are to be multiplied by a suitable factor in order to follow the asymptotic behaviour (2.3) *a posteriori*; this invariance is in fact the freedom one has to rescale the time coordinate or the z coordinate. In order to avoid the *a posteriori* normalisation, we perform the following change of functions

$$a(r) = \frac{r^2}{\ell^2} \hat{a}(r), \quad b(r) = \frac{r^2}{\ell^2} \hat{b}(r), \quad f(r) = \frac{r^2}{\ell^2} \hat{f}(r). \quad (2.4)$$

The advantage of this change of functions is that the hatted functions normalisation is fixed once for all, independently of the number of dimension and of the AdS radius by imposing $\hat{a} \approx \hat{b} \rightarrow 1$ as $r \rightarrow \infty$. This normalisation doesn't require an *a posteriori* rescaling of the solution.

Note that the z coordinate ranges from 0 to some length L . The value of L will be fixed to be the critical length where the linear perturbations are static. This precise value of L is provided by the stability analysis coming from the first order in perturbation theory [28].

It must be stressed that the equations resulting from the ansatz (2.2) require non-trivial regularity conditions for the correction functions equations. However, going in a 'conformal-like' gauge,

$$ds^2 = \frac{g(\tilde{r})}{\ell^2} \tilde{b}(\tilde{r}) e^{2A(r,z)} dt^2 + e^{2B(r,z)} \left(\frac{\ell^2 dr^2}{g(\tilde{r})^2 \tilde{f}(\tilde{r})} + \frac{g(\tilde{r}) L^2 \tilde{a}(\tilde{r}) dz^2}{\ell^2} \right) + g(\tilde{r}) e^{2C(r,z)} d\Omega_{d-3}^2, \quad (2.5)$$

where $g(r) = \tilde{r}^2 + r_h^2$, r_h being the horizon radius, leads to much simpler regularity conditions. Note that we rescaled the z coordinate such that $z \in [0, 1]$ by factorising the length L .

The relation between (2.2) and (2.5) is given by

$$\tilde{r}^2 + r_h^2 = r^2, \quad \tilde{a}(\tilde{r}) = a(r), \quad \tilde{b}(\tilde{r}) = b(r), \quad \tilde{f}(\tilde{r}) = f(r). \quad (2.6)$$

2.1 The equations and boundary conditions

From now on, the a, b, f functions and r coordinate will refer to the functions and variables of the conformal gauge (2.5). The Einstein equations with the ansatz (2.5) lead to the following set of coupled partial differential equations:

$$\begin{aligned} & - \left(\frac{(-1+d)e^{2B}r^2}{fg^2} \right) - \frac{b'}{2rb} + \frac{a'b'}{4ab} - \frac{b'^2}{4b^2} + \frac{b'f'}{4bf} - \frac{g'}{2rg} + \frac{a'g'}{4ag} + \frac{(2+d)b'g'}{4bg} + \frac{f'g'}{4fg} \\ & + \frac{(-1+d)g'^2}{4g^2} + \frac{b''}{2b} + \frac{g''}{2g} + \frac{\ell^4 r^2 A^{(0,1)2}}{L^2 a f g^3} + \frac{(-3+d)\ell^4 r^2 A^{(0,1)} C^{(0,1)}}{L^2 a f g^3} + \frac{\ell^4 r^2 A^{(0,2)}}{L^2 a f g^3} - \frac{A^{(1,0)}}{r} \\ & + \frac{a'A^{(1,0)}}{2a} + \frac{b'A^{(1,0)}}{b} + \frac{f'A^{(1,0)}}{2f} + \frac{(2+d)g'A^{(1,0)}}{2g} + A^{(1,0)2} + \frac{(-3+d)b'C^{(1,0)}}{2b} \\ & + \frac{(-3+d)g'C^{(1,0)}}{2g} + (-3+d)A^{(1,0)}C^{(1,0)} + A^{(2,0)} = 0 \end{aligned} \quad (2.7)$$

$$\begin{aligned} & \frac{(-4+d)(-3+d)e^{2B-2C}\ell^2 r^2}{2fg^3} + \frac{(-4+d)(-1+d)e^{2B}r^2}{2fg^2} - \frac{a'}{2ra} - \frac{a'^2}{4a^2} + \frac{a'f'}{4af} - \frac{g'}{2rg} + \frac{a'g'}{ag} \\ & - \frac{(-3+d)b'g'}{4bg} + \frac{f'g'}{4fg} - \frac{(-4+d)(-1+d)g'^2}{8g^2} + \frac{a''}{2a} + \frac{g''}{2g} - \frac{(-3+d)\ell^4 r^2 A^{(0,1)} C^{(0,1)}}{L^2 a f g^3} \\ & - \frac{(-4+d)(-3+d)\ell^4 r^2 C^{(0,1)2}}{2L^2 a f g^3} + \frac{\ell^4 r^2 B^{(0,2)}}{L^2 a f g^3} - \frac{(-3+d)g'A^{(1,0)}}{2g} - \frac{B^{(1,0)}}{r} + \frac{a'B^{(1,0)}}{2a} \\ & + \frac{f'B^{(1,0)}}{2f} + \frac{3g'B^{(1,0)}}{2g} - \frac{(-3+d)b'C^{(1,0)}}{2b} - \frac{(-3+d)^2 g'C^{(1,0)}}{2g} - (-3+d)A^{(1,0)}C^{(1,0)} \\ & - \frac{(-4+d)(-3+d)C^{(1,0)2}}{2} + B^{(2,0)} = 0 \end{aligned} \quad (2.8)$$

$$\begin{aligned} & - \left(\frac{(-4+d)e^{2B-2C}\ell^2 r^2}{fg^3} \right) - \frac{(-1+d)e^{2B}r^2}{fg^2} - \frac{g'}{2rg} + \frac{a'g'}{4ag} + \frac{b'g'}{4bg} + \frac{f'g'}{4fg} + \frac{(-1+d)g'^2}{4g^2} \\ & + \frac{g''}{2g} + \frac{\ell^4 r^2 A^{(0,1)} C^{(0,1)}}{L^2 a f g^3} + \frac{(-3+d)\ell^4 r^2 C^{(0,1)2}}{L^2 a f g^3} + \frac{\ell^4 r^2 C^{(0,2)}}{L^2 a f g^3} + \frac{g'A^{(1,0)}}{2g} - \frac{C^{(1,0)}}{r} + \frac{a'C^{(1,0)}}{2a} \\ & + \frac{b'C^{(1,0)}}{2b} + \frac{f'C^{(1,0)}}{2f} + \frac{(-1+d)g'C^{(1,0)}}{g} + A^{(1,0)}C^{(1,0)} + (-3+d)C^{(1,0)2} + C^{(2,0)} = 0 \end{aligned} \quad (2.9)$$

where the exponent (m, n) denotes the m^{th} derivative with respect to r and n^{th} derivative with respect to z and the primes denote the derivative with respect to r for functions depending on r only.

The system of partial differential equations above is supplemented by the following boundary conditions:

$$\begin{aligned}
 A(r, z) &= B(r, z) = C(r, z) = 0 \quad \text{for } r \rightarrow \infty, \\
 \partial_z A(r, z) &= \partial_z B(r, z) = \partial_z C(r, z) = 0 \quad \text{for } z = 0, 1, \\
 \partial_r A(r, z) &= \partial_r C(r, z) = 0, \quad B(r, z) = A(r, z) + d_0 \quad \text{for } r = 0,
 \end{aligned} \tag{2.10}$$

where d_0 is a real deformation parameter related to the variation of the temperature along the non uniform branch. The first condition ensures the fact that A, B, C are *corrections* and thus vanish at infinity, the second conditions imposes periodicity along the z coordinate while the last three conditions are the regularity conditions.

3 Properties of the solution

3.1 Asymptotic and near horizon behaviour

The asymptotic behaviour of the background functions is given in [20]. Since we factorise the $\frac{r^2}{\ell^2}$ term, the asymptotic developpement in [20] is of course to be divided by r^2/ℓ^2 , leading to

$$\begin{aligned}
 a &= 1 + \sum_{i=0}^{\lfloor \frac{d-4}{2} \rfloor} a_i \left(\frac{\ell}{r}\right)^{2i+2} + c_z \left(\frac{\ell}{r}\right)^{d-1} + \sum_k \delta_{d,2k+1} \xi \log \frac{r}{\ell} \left(\frac{\ell}{r}\right)^{d-1} + \mathcal{O}\left(\frac{\ell}{r}\right)^d, \\
 b &= 1 + \sum_{i=0}^{\lfloor \frac{d-4}{2} \rfloor} a_i \left(\frac{\ell}{r}\right)^{2i+2} + c_t \left(\frac{\ell}{r}\right)^{d-1} + \sum_k \delta_{d,2k+1} \xi \log \frac{r}{\ell} \left(\frac{\ell}{r}\right)^{d-1} + \mathcal{O}\left(\frac{\ell}{r}\right)^d, \\
 f &= 1 + \sum_{i=0}^{\lfloor \frac{d-4}{2} \rfloor} f_i \left(\frac{\ell}{r}\right)^{2i+2} + (c_t + c_z + c_0) \left(\frac{\ell}{r}\right)^{d-1} + \sum_k \delta_{d,2k+1} \xi \log \frac{r}{\ell} \left(\frac{\ell}{r}\right)^{d-1} + \mathcal{O}\left(\frac{\ell}{r}\right)^d,
 \end{aligned} \tag{3.1}$$

where δ is the Kronecker delta, $\lfloor X \rfloor$ stands for the floor integer value of X , c_t, c_z are parameters to be determined numerically and a_i, f_i, c_0, ξ depend on d and are given in [20].

It has been argued in [6, 29] that the modes of A, B, C in a perturbation theory decay as $\left(\frac{\ell}{r}\right)^{d-1}$ in all order of the perturbation theory. Since the non perturbative solution is a combination of these modes, the leading order should also decay as

$$\begin{aligned}
 A(r, z) &= A_1 \left(\frac{\ell}{r}\right)^{d-1} + \mathcal{O}\left(\frac{\ell}{r}\right)^d, & B(r, z) &= B_1 \left(\frac{\ell}{r}\right)^{d-1} + \mathcal{O}\left(\frac{\ell}{r}\right)^d, \\
 C(r, z) &= C_1 \left(\frac{\ell}{r}\right)^{d-1} + \mathcal{O}\left(\frac{\ell}{r}\right)^d.
 \end{aligned} \tag{3.2}$$

for some $A_1, B_1, C_1 \in \mathbb{R}$. The decay (3.2) is confirmed by direct computations.

Close to the horizon, where $r \approx r_h$, the background fields obey the following series developpement to the leading order

$$\begin{aligned}
 a(r) &= a_0 + \mathcal{O}(r - r_h), & b(r) &= b_1(r - r_h) + \mathcal{O}(r - r_h)^2, \\
 f(r) &= \frac{1}{r_h^3} ((d-1)r_h^2 + (d-4)\ell^2)(r - r_h) + \mathcal{O}(r - r_h)^2,
 \end{aligned} \tag{3.3}$$

where a_0, b_1 are real numbers fixed by the background boundary condition $a, b, f \rightarrow 1$ for $r \rightarrow \infty$. Subleading corrections can be found in [20], where the coefficients are to be multiplied by ℓ^2/r_h^2 due to the change of functions (2.4).

3.2 Thermodynamical properties

The temperature, mass, entropy and tension of the background is given by [20]:

$$\begin{aligned}
 T_U &= \frac{1}{4\pi} \sqrt{\frac{b_1}{r_h \ell^2} ((d-1)r_h^2 + (d-4)\ell^2)}, & S_U &= \frac{r_h^{d-1}}{4\ell^2} a_0 L V_{d-3}, \\
 M_U &= \frac{L \ell^{d-4} V_{d-3}}{16\pi} (c_z - (d-2)c_t) + M_U^0, & \mathcal{T}_U &= \frac{\ell^{d-4} V_{d-3}}{16\pi} ((d-2)c_z - c_t) + \mathcal{T}_U^0,
 \end{aligned} \tag{3.4}$$

where $M_U^0 = \mathcal{T}_U^0 L$ are Casimir-like terms occurring in odd dimensions [20].

The thermodynamical properties of the background has been studied in [20] and the thermodynamical properties of the non uniform strings has been studied perturbatively in [6, 29]. The correction functions provide corrections on the thermodynamical quantities computed from the background (as the name correction function suggests). The quantities computed on the background solution are denoted with a subscript U , while the quantities referring to the entire non uniform solution will be denoted with a subscript NU . The corrections on the thermodynamical quantities are then given by

$$\begin{aligned}
 T_{NU} &= e^{-d_0} T_U, \\
 S_{NU} &= \frac{S_U}{L} \int_0^1 L e^{B(r_h, z) + (d-3)C(r_h, z)} dz,
 \end{aligned} \tag{3.5}$$

where T_{NU} and S_{NU} are the Hawking temperature (resp. the entropy) of the non uniform string. Note that the first relation makes the role of d_0 clearer: once the background is set, the parameter d_0 allows to slide along the non uniform branch in a temperature-entropy phase diagram.

These quantities are defined at the horizon and are relatively easy to compute from a technical point of view. The asymptotic quantities are given by

$$\begin{aligned}
 M_{NU} &= M_U \left(1 + \frac{\delta M}{M_U} \right), \\
 \mathcal{T}_{NU} &= \mathcal{T}_U \left(1 + \frac{\delta \mathcal{T}}{\mathcal{T}_U} \right).
 \end{aligned} \tag{3.6}$$

where $\delta M, \delta \mathcal{T}$ are functions of the parameters A_1, B_1, C_1 appearing in (3.2) and c_t, c_z, c_0 . Note that at the boundary spacetime, ∂_z and ∂_t are both Killing vectors, so the definition of the mass and tension as functions of the asymptotic parameters [20] remain the same. Note also that the decay (3.2) combined with the asymptotic background solution gives contribution of order $r^{-(d-3)}$ to g_{tt}, g_{zz} and leads to correction on c_t, c_z , thus on the mass and tension; another decay of the corrections would then be physically rejectable.

However, the quantities c_t, c_z and worse, A_1, B_1, C_1 are difficult to compute, first because of the sharpness of the decay and second because the determination of these quantities

is plagued by numerical noise. In order to avoid this problem, we were able to integrate the first law of black holes in order to compute the masses (of the uniform solution *and* of the non uniform solution). Assuming that the Smarr relation $M + L\mathcal{T} = T_H S$, holds for the non uniform solution, we were able to compute the corrections on the tension for the non uniform solution.

The procedure to integrate the first law is performed in two steps, first at the background level, then for the non uniform solution. At the background level, given the mass per unit length for a solution, say with $\ell = 1, r_h = r_h^*$ and given the entropy and temperature as a function of r_h , we can integrate the first law of black objects according to

$$\frac{M_U(r_h)}{L} = \frac{M_U(r_h^*)}{L} + \frac{1}{L} \int_{r_h^*}^{r_h} T_H(r_h) \frac{\partial S_U(r_h)}{\partial r_h} dr_h, \quad (3.7)$$

where we have assumed L to be a constant. This assumption doesn't spoil generality since L plays a spectator role in the background.

The tension is computed using the Smarr relation once the mass is known:

$$\mathcal{T}_U = \frac{M_U}{L} - T_H^U \frac{S_U}{L}. \quad (3.8)$$

The integration of the first law for the non uniform solution is more straightforward since we already compute *corrections* as a function of the parameter d_0 . The procedure to integrate the first law seems more natural in this case; given a uniform black string with horizon radius r_h , the non uniform black string with a given value of d_0 emanating from this uniform black string has a mass given by

$$M_{NU}(r_h) = M_U(r_h) + \int_0^{d_0} T_H^{NU} \frac{\delta S}{\delta d_0} d d_0 = M_U(r_h) + T_H^U \int_0^{d_0} e^{-d_0} \frac{\delta S}{\delta d_0} d d_0, \quad (3.9)$$

since the length of the non uniform black string is fixed and where $\delta S/\delta d_0$ is of course to be evaluated numerically.

Once the mass of the non uniform string for the given value of d_0 computed, it is again straightforward to obtain the tension for the corresponding value of d_0 , assuming the relation:

$$\mathcal{T}_{NU} = \frac{T_H^{NU} S_{NU}}{L} - \frac{M_{NU}}{L}. \quad (3.10)$$

3.2.1 Note on the tension of the background

A regular solution in the asymptotically locally flat spacetime would be the $d - 1$ dimensional Minkowski spacetime times a circle, $\mathcal{M}_{d-1} \times S^1$. Of course, this solution has zero mass and zero tension. The d dimensional uniform black string in the flat case is characterised by a positive mass and tension. However, the AdS counterpart of the regular solution is characterised by a positive non vanishing mass [20], a finite temperature and a vanishing entropy. Thus, using the Smarr relation, the tension of the regular solution is negative:

$$M + \mathcal{T}L = 0 \Rightarrow \mathcal{T} = -\frac{M}{L}. \quad (3.11)$$

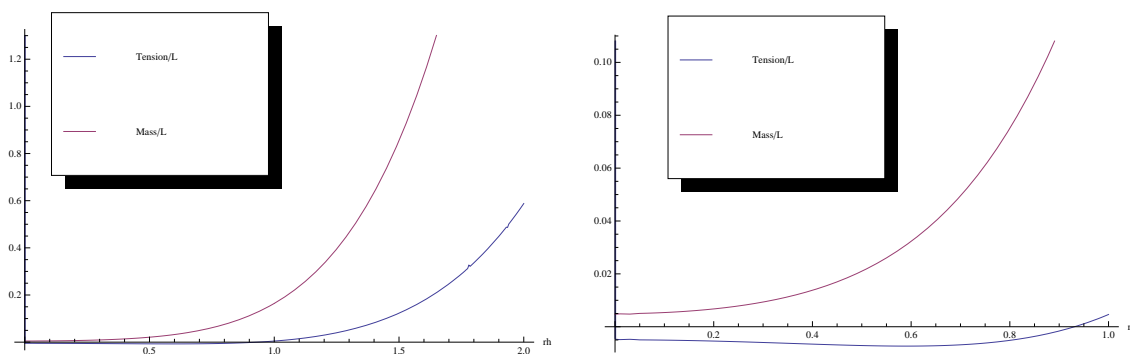


Figure 1. The mass and tension per unit length for the uniform black string in $d = 6, \ell = 1$. Large values of r_h exhibit a positive tension while smaller values of r_h have negative tension.

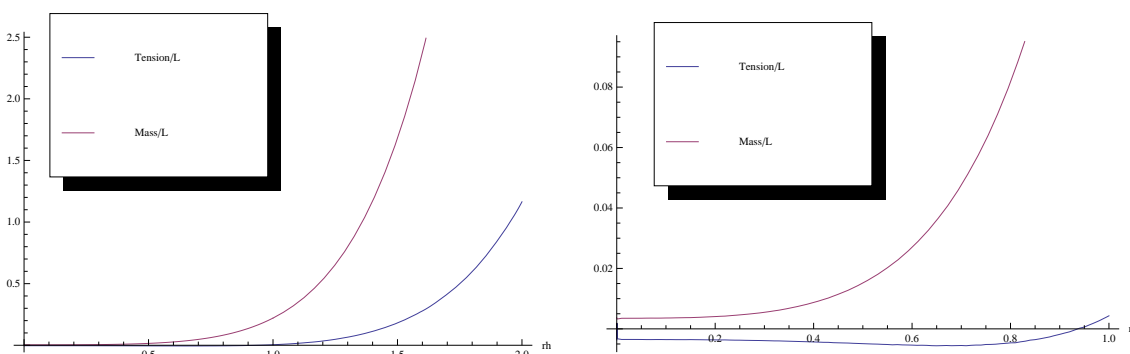


Figure 2. This figure is the same as 1, except that here, $d = 7$.

It turns out that the black strings with small values of r_h also have negative tension, these black strings are continuously connected to the AdS regular solution. This is confirmed by a numerical analysis as shown in figure 1 and 2 for $d = 6, \ell = 1$ and $d = 7, \ell = 1$. These values were obtained using the definition of the mass and tension as a function of c_t, c_z in [20] (in units where the Newton constant is set to one). Note however that the regular solution ($r_h = 0$) has a vanishing total energy since $E = M + \mathcal{T}L = 0$, where E denotes the total energy [9].

The negative tension seems to be an effect of the AdS curvature, since this phenomenon doesn't appear in the asymptotically locally flat uniform string problem. This might be interpreted as follows: on the one hand, a negative cosmological is characterised by a positive constant pressure. On the other hand, tension on strings typically acts oppositely to the pressure. In this picture, when the gravitational tension is too small, the pressure from AdS dominates, inducing a negative tension. If the string is massive enough, the gravitational tension is higher and then dominates the pressure. The effect on the sign of the tension then results from a balance between the AdS pressure and the gravitational tension.

We checked the sign of the tension numerically for $d = 6, 7$ but we expect this property to hold for every value of the number of dimension.

3.2.2 Entropy difference at same mass

In the asymptotically locally flat spacetime case, it is possible to evaluate the entropy difference at same mass between the non uniform and uniform string using the following relation (see for example [26, 30]):

$$\frac{S_{\text{Non uniform}}}{S_{\text{Uniform}}} = 1 + \sigma_1 \lambda^2 + \sigma_2 \lambda^4 + \dots, \tag{3.12}$$

where σ_1, σ_2 are functions of the corrections on the temperature, entropy, mass and tensions computed in a perturbative approach and λ is the deformation parameter in this perturbative approach [30].

This relation permitted to establish the existence of a critical dimension above which the order of the phase transition between the uniform black string and the non uniform one changes from one to higher.

However, in order to establish this kind of relation, one uses the mass dependence of the entropy of the $(d - 1)$ dimensional Tangherlini black hole, since the latter are foliations at fixed z of the uniform black string (in asymptotically locally flat spacetime). However in asymptotically locally AdS, one cannot rely on the same argument since the solution for the uniform black string is numerical and the relation between the mass and entropy of the constant z foliation is not clear in this case; so one has to rely on numerical computations.

At fixed value of the AdS length, from a numerical point of view, the mass and entropy in the uniform phase are functions of $r_h, M(r_h)$ (resp. $S(r_h)$) while the corrections on the mass and on the entropy along the non uniform string are functions of r_h and d_0 , say $\delta M(r_h, d_0)$ (resp. $\delta S(r_h, d_0)$). So one can compute the variation of the mass along the non uniform branch and find the value of r_h , say r_h^* , such that the uniform string has the same mass. Once this value is found, it is trivial to compute the difference between the entropy of the non uniform phase emerging from r_h and with a given value of d_0 and the entropy of the uniform string with horizon radius r_h^* :

$$r_h^* \text{ is such that } M(r_h) + \delta M(r_h, d_0) = M(r_h^*). \tag{3.13}$$

Then, the relative entropy difference between the non uniform string with horizon radius r_h and the uniform string of same mass is a function of d_0 and is given by

$$\frac{S_{\text{Non uniform}} - S_{\text{Uniform}}}{S_{\text{Uniform}}} = \frac{(S(r_h) + \delta S(r_h, d_0)) - S(r_h^*)}{S(r_h^*)}. \tag{3.14}$$

We will use the relation (3.14) in order to investigate the order of the phase transition. If the entropy of the uniform string is higher than the entropy of the non uniform one for the same value of the mass in a small deformation regime, the order of the phase transition is 1. If the entropy of the non uniform string increases at the same mass, the order is higher than one and the transition is smooth. In other words if the quantity evaluated from (3.14) is negative, the order of the phase transition is one, if the quantity is positive, the order is higher.

We will present the result of this investigation in the case where $d = 7$ in the next section.

3.3 Relation between deformation and thermodynamics

The order parameter for the transition from the uniform string phase to the non uniform string is defined by [26]

$$\lambda = \frac{R_{\text{Max}}}{R_{\text{Min}}} - 1, \tag{3.15}$$

where R_{Min} (resp. R_{Max}) is the maximum (resp. minimum) of the areal radius, defined as $R = r e^{C(r,z)}$.

The parameter λ measures the deformation of the solution; the uniform string is characterised by $\lambda = 0$, while the hypothetical localised black hole phase has an infinite deformation parameter. It has been argued in [31] that using the mass and tension is more relevant in the construction of the phase diagram; the mass and tension remain finite in the three phases (uniform, non uniform and localised black hole/string). However, the mass and tension are defined at the boundary spacetime and are difficult to compute from a technical point of view. An alternative is to study the phase diagram in T_H, S coordinates; these quantities are defined at the horizon of the black objects, and are easier to compute.

As already noted in [6], λ increases with d_0 when the cosmological constant increases. This can be understood as follows: the deformation is related to the variation of entropy and d_0 is related to the variation of the temperature. Moreover the rate of variation of λ with d_0 is proportional to the variation of the entropy with respect to the temperature along the non uniform phase. In order to prove this explanation, one should recall that for small values of λ , the non uniform string is well described by the first order of perturbation theory (where $C(r, z) \approx \epsilon C_1(r) \cos 2\pi z$, ϵ being the small parameter of the perturbation). In this approximation, one can fix the value of ϵ for a given value of λ :

$$\lambda = \frac{e^{\epsilon C_1(r_h)} + \mathcal{O}(\epsilon)^2}{e^{-\epsilon C_1(r_h)} + \mathcal{O}(\epsilon)^2} - 1 = 2\epsilon C_1(r_h) + \mathcal{O}(\epsilon)^2. \tag{3.16}$$

Recall that from the boundary conditions of the first order in perturbation theory [28], $C_1(r_h) = 1$, leading to $\epsilon = \lambda/2$.

On the other hand, the variation of the temperature along the non uniform branch is given by $T_H^{NU} = T_U e^{-d_0}$; a variation δd_0 in d_0 gives rise to a variation of the temperature δT_H^{NU} in the non uniform phase according to $\delta T_H^{NU} = -T_H^U e^{-d_0} \delta d_0$. Moreover, the correction on the entropy in the perturbative approach appears in order ϵ^2 in the perturbation theory; in other words, is proportional to λ^2 .

So, increasing d_0 from 0 to $d_0 \ll 1$ implies a variation of λ of 0 to $\lambda \ll 1$ and

$$\frac{\delta S}{\delta T_H^{NU}} = -T_H^{NU} \frac{\lambda^2}{d_0} \approx -T_H^U \frac{\lambda^2}{d_0}. \tag{3.17}$$

In other words, the value of λ corresponding to a (small) value of d_0 is a monotonic negative function of $\frac{\delta S}{\delta T_H^{NU}}$.

To sum up, if $\frac{\delta S}{\delta T_H^{NU}}$ is large and negative, λ will be large and positive and conversely. This was noticed but not explained in [6] and is shown for $d = 7$ for two values of Λ in figure 3.

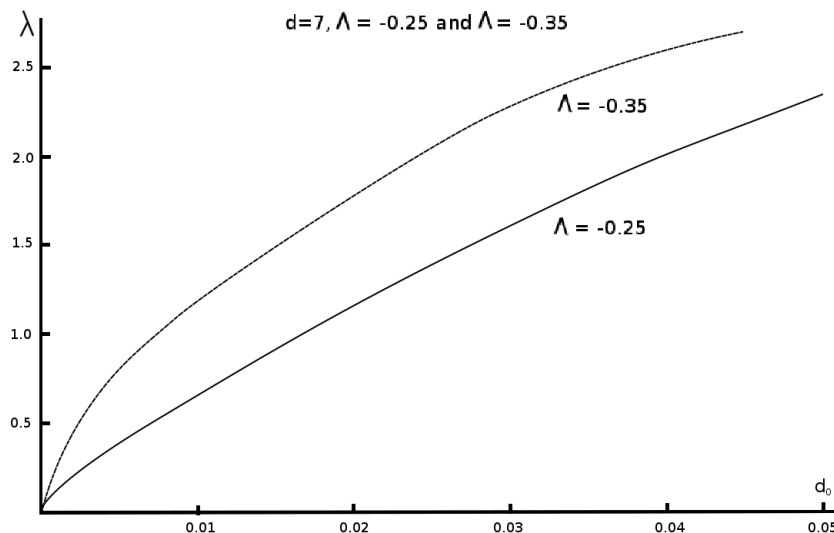


Figure 3. The deformation increases faster for larger absolute values of the cosmological constant, this is to be related to the increase of $\delta S/\delta T$ with Λ (large absolute values of Λ at fixed r_h correspond to large values of r_h at fixed Λ from the scaling relations in [20]).

4 Numerical results and discussion

4.1 Numerical procedure

In order to integrate the equations (2.7), (2.8), (2.9) we first integrate the background equations and the first order of perturbation theory numerically in order to set the background and to fix the length of the static perturbation, where the non uniform phase should emerge. The integration is carried out numerically with the solver *Colsys* [32]. Then, we integrate the system of partial differential equation numerically, using the solver *FidiSol* [33] with the suitable boundary conditions described in the previous section. Let us mention the fact that the numerical procedure is plagued by many difficulties. First, the background functions are numerical and it is not clear how the numerical errors on the background influences the errors on the corrections.

Moreover, one cannot start from the trivial solution $A = B = C = 0$; Fidisol needs an initial guess sufficiently close to the solution. In practise, it is sufficient to use the solution to the first order in perturbation theory as a starting profile.

Let us also mention that if the parameters are not chosen in a suitable manner, the solver produces a uniform solution which seems to be a 'correction' to the uniform string leading to a uniform string with a different temperature. Of course, we are not interested in this kind of solutions. Moreover, it happened that the solution is a kind of combination of this uniform correction and some non uniform correction, leading to a decrease of the deformation along the non uniform string, even for small values of the cosmological constant (or equivalently for small values of the horizon radius at fixed AdS radius) where we expect the pattern of the solution to be similar to the case where the cosmological constant vanishes.

The region of larger horizon radius (but still in the unstable phase of the uniform black string) is even more difficult to investigate: we expect a transition from thermodynamically unstable non uniform black strings to thermodynamically stable non uniform black strings following the arguments presented in [6]. These thermodynamically stable (resp. unstable) non uniform black string were referred to as small-short (resp. small-long) non uniform black string since they are characterised by a small horizon radius and a short (resp large) critical length (compared to the AdS curvature). The boundary separating these two phases, small-short and small-long black string, is characterised by $\delta S/\delta T_H \rightarrow \infty$. As we argued in section 3.3 the deformation would become extremely high even for very small values of d_0 (thus of δT_H). Of course, this extreme behaviour is not manageable from a technical point of view and sets the limit of our numerical investigation.

However, we were able to find some hints for the existence of non uniform solutions in a neighbourhood of this region, but these solutions have large numerical errors and it is not reliable to perform quantitative computations.

4.2 Numerical solutions

For the reason described above, we restrict the study of the AdS non uniform black string to the region of the parameter space where $r_h \ll \ell$ and $L \ll \ell$.

Despite of all the numerical difficulties, we were able to construct several solutions for various values of d , Λ and d_0 . In every case considered, the deformation is an increasing function of d_0 and we systematically found extremely deformed solution, providing strong indications for the existence of the AdS localised black hole phase. In figure 4, we present embeddings of the event horizon in a 3 dimensional cartesian space; the strings is almost uniform for small values of d_0 and looks like a sphere touching the borders for larger values of d_0 .

4.3 Horizon radius depending critical dimension

We have evaluated the entropy difference between the non uniform black string and the uniform black string at same mass and found that above some critical values of the horizon radius, the uniform black string is more entropic than the non uniform one while for larger values of r_h , the non uniform black string becomes more entropic. In other words, the phase transition can be of first order or of higher order for the same value of the dimension, depending on the value of the horizon radius. We present a plot of the variation of the entropy difference as a function of d_0 for $d = 7, \ell = 1$ in figure 5 for various values of the horizon radius. Small values have a negative slope while larger value acquire a positive slope, changing the order of the phase transition.

It should be noted that the value of d_0 in figure 5 is very small, and thus the non uniform solutions are weakly deformed. This regime of small deformation is the regime where the solutions to the non linear set of partial differential equations are the closest to the perturbative solutions. Moreover, the variation of entropy and mass are computed according to the procedure described in section 3.2.2, involving a comparison between the entropy of the background and of the non uniform solution. So on the one hand, we have the background uniform solution which is computed very accurately (the relative error is of

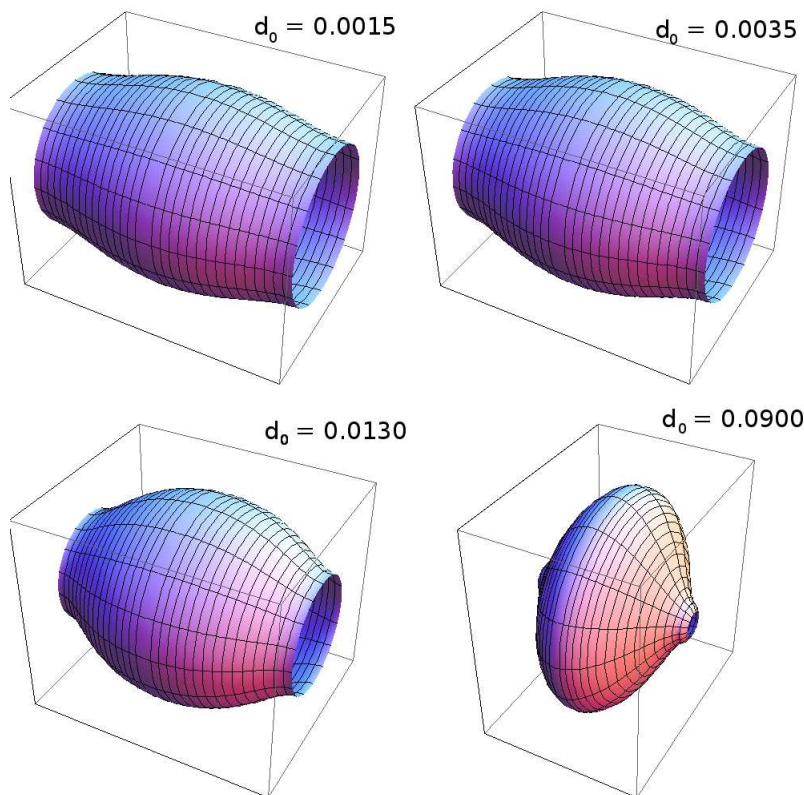


Figure 4. Embedding of the horizon of non uniform black strings with $d = 7$, $\Lambda = -0.25$ in a 3 dimensional euclidean space. The horizon looks like a black hole horizon as the deformation increases.

the order 10^{-5}) and on the other hand, the non uniform solution is in the best controlled region, where the relative root mean square error on the solution is of the order 10^{-2} . Moreover, the quantities involved in the entropy difference computation are extracted from the horizon data, which is the more reliable region.

Figure 5 thus provides strong indication for the existence of a r_h -depending critical dimension and we expect this property to hold for all values of the dimension.

5 Conclusion

We have constructed the non uniform asymptotically locally AdS black string for various values of the number of dimensions (typically 5, 6, 7, 8). We were able to numerically integrate the system of coupled partial differential equations despite of the numerical difficulties and found numerical evidences for the existence of the non uniform black string. Moreover, the variation of the deformation of the solutions with respect to the temperature is closely related to the variation of the entropy with temperature in the small deformation region. This of course makes the investigation of the small-long AdS non uniform black string [6] very difficult from a technical point of view. However, the small short AdS black strings are easier to deal with and we found strong evidence for the existence of a AdS

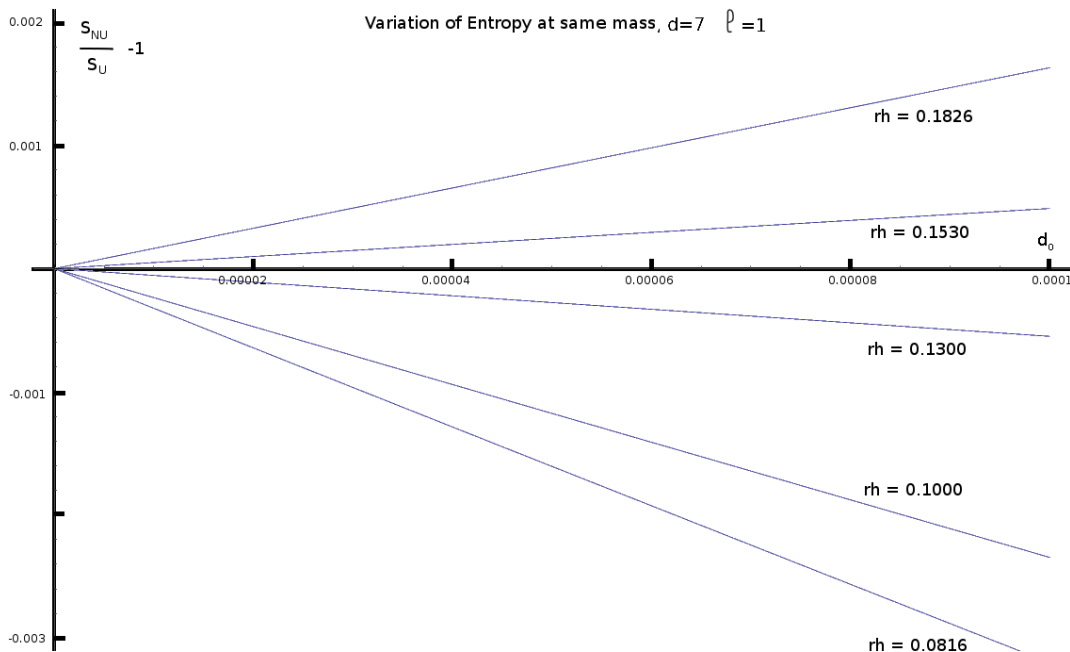


Figure 5. The relative entropy difference between the non uniform phase and uniform phase at same mass as a function of d_0 for $d = 7, \ell = 1$ and various value of r_h .

localised black hole phase in the strongly deformed limit of the non uniform black string for the dimensions we considered. From a qualitative point of view, the solutions behave similarly in all the dimensions we considered, so we expect all these properties to hold for an arbitrary number of dimensions.

Moreover, it might have been expected from the fact that the variation of the entropy with respect to the variation of the temperature increases with r_h , that the critical dimension for the order of the phase transition of the uniform - non uniform black string depends on r_h , at fixed AdS radius. It seems indeed that this is the case. We numerically compared the entropy of non uniform black strings and uniform strings with same mass for $d = 7$ and found that above some value of r_h ($r_h \approx 0.14$ for $\ell = 1$) the entropy along the non uniform branch increases quicker than the entropy of the uniform string with same mass. In other words, the transition becomes smooth, thus of order higher than one. Once again, we expect this property to hold for other numbers of dimensions, at least for $d < d^*$ where d^* is the critical dimension reported in [30].

We also argued that the asymptotically locally AdS regular solution has a negative tension and a positive mass. It was already noticed that the mass was positive in [20], but we stress the fact that the tension is negative. The tension is also negative for uniform black strings with small values of the horizon radius (compared to the AdS radius). However, the total energy for the regular solution is null and positive for the small AdS black strings. We interpret this property as resulting from a balance between the gravitational tension of the string (increasing with the mass) and the pressure of the AdS space; if the gravitational tension is too small, the AdS pressure dominates and the string is not bound (the tension is interpreted as a binding energy when multiplied by the length of the z coordinate). This

might have an important consequence on the existence of an hypothetical AdS non rotating thin black ring, where the gravitational attraction of the ring would be balanced by the negative tension of the string portions, since thin black rings are locally well described by black strings.²

Note that the various new properties we exposed here should influence the dual *CFT* in the *AdS/CFT* context.

Acknowledgments

I would like to gratefully acknowledge Yves Brihaye and Eugen Radu for very useful discussions and wise advices. I also thank Fabien Buisseret and Georges Kohnen.

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²I thank the unknown referee for pointing me out this issue and application of the negative tension phenomenon.

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